Operations with rational algebraic expressions

Greatest common divisor, and lowest common denominator polynomial

GCD polynomial P and Q is a polynomial **D**, which has the highest level among polynomials, which are dividers polynomial P and Q.

LCD polynomial P and Q is a polynomial **S**, who has the lowest level among polynomials, which are divisible polynomials P and Q.

Example 1: Find S and D for polynomials:

$$P(x) = x^2 - 4$$
$$Q(x) = x^2 - x - 2$$
$$R(x) = x^2 - 3x + 2$$

First, we must each of them apart on the facts (of course, the use of the procedure mentioned in the previous chapter).

$$P(x) = x^2 - 4 = x^2 - 2^2 = (x-2)(x+2)$$
$$Q(x) = x^2 - x - 2 = x^2 - 2x + x - 2 = x(x-2) + 1(x-2) = (x-2)(x+1)$$
$$R(x) = x^2 - 3x + 2 = x^2 - x - 2x + 2 = x(x-1) - 2(x-1) = (x-1)(x-2)$$

D is actually “one”, which it has in each of polynomials.

It is obvious x-2. So:

GCD= x-2 or **D = x-2**

S is “Union”. It must be divisible by all three polynomials. So:

S = (x-2)(x+2)(x-1)(x+1)

Example 2: Find S and D for polynomials:

$$P = a^2 - ab$$
$$Q = a^2 - b^2$$
$$R = a^2 - 2ab + b^2$$

Solution:

$$P = a^2 - ab = a(a-b)$$
$$Q = a^2 - b^2 = (a-b)(a+b)$$
$$R = a^2 - 2ab + b^2 = (a-b)^2$$

D = $(a-b) \rightarrow$ because it has all three

S = $a(a-b)^2(a+b) \rightarrow$ divisible by all three

Example 3: Find S and D for polynomials:

$$A = x^2 - xy$$
$$B = xy + y^2$$

Solution:

$$\begin{aligned} A &= x(x-y) \\ B &= y(x+y) \end{aligned} \quad \Rightarrow \quad \text{S} = \; xy(x-y)(x+y)$$

What will be D? There are no factors that are contained in A and B. In such a situation D = 1, and we say that

polynomials are the mutual primes.

Example 4: Find **S and D** for polynomials:

$$9a + 15 =$$
$$36a^2 - 100 =$$
$$-9a^2 + 30a - 25 =$$

Solution:

$$9a + 15 = 3(3a+5)$$
$$36a^2 - 100 = 4(9a^2 - 25) = 4(3a-5)(3a+5)$$
$$-9a^2 + 30a - 25 = -(9a^2 - 30a + 25) = -(3a-5)^2$$

$\text{S} = -12(3a+5)(3a-5)^2$

D = 1

Example 5: Find S and D for polynomials:

$$4a^2 + 4ab + b^2 = (2a+b)^2$$
$$4a^2 - b^2 = (2a-b)(2a+b)$$
$$8a^3 + b = (2a)^3 + b^3 = (2a+b)(4a^2 - 2ab + b^2)$$

Solution:

$$4a^2 + 4ab + b^2 = (2a+b)^2$$
$$4a^2 - b^2 = (2a-b)(2a+b)$$
$$8a^3 + b = (2a)^3 + b^3 = (2a+b)(4a^2 - 2ab + b^2)$$

S $= (2a+b)^2(2a-b)(4a^2 - 2ab + b^2)$

D= $2a$+b

Example 6: Find S for polynomials:

$$3x^3-12x^2+12x=$$
$$5x^4+20x^3+20x^2=$$
$$3nx^2-12n=$$

Solution:

$$3x^3-12x^2+12x=3x(x^2-4x+4)=3x(x-2)^2$$
$$5x^4+20x^3+20x^2=5x^2(x^2+4x+4)=5x^2(x+2)^2$$
$$3nx^2-12n=3n(x^2-4)=3n(x-2)(x+2)$$

$$\mathrm{S}=15nx^2(x-2)^2(x+2)^2$$

Example 7: Find S for polynomials:

$$2a^4-2=$$
$$a^3+a^2+a+1=$$
$$a^3-a^2+a+1=$$

Solution:

$$2a^4-2=2(a^4-1)=2(a^2-1)(a^2+1)=2(a-1)(a+1)(a^2+1)$$
$$a^3+a^2+a+1=a^2(a+1)+1(a+1)=(a+1)(a^2+1)$$
$$a^3-a^2+a+1=a^2(a-1)+1(a-1)=(a-1)(a^2+1)$$

$$\mathrm{S}=2(a-1)(a+1)(a^2+1)$$

How to use S?

1) **Simplify expression:** $\frac{a}{ab-b^2}+\frac{b}{a^2-ab}-\frac{a+b}{ab}=$

$\frac{a}{ab-b^2}+\frac{b}{a^2-ab}-\frac{a+b}{ab}=$ first every denominator apart on factors =

$\frac{a}{b(a-b)}+\frac{b}{a(a-b)}-\frac{a+b}{ab}=$ then find S, it is $ab(a-b)$,make fractional expansion.

$$=\frac{a\cdot a+b\cdot b-(a+b)(a-b)}{ab(a-b)}=$$

$$=\frac{a^2+b^2-(a^2-b^2)}{ab(a-b)}=\frac{a^2+b^2-a^2+b^2}{ab(a-b)}=\frac{2b^2}{ab(a-b)}=\frac{2b^{\not 2}}{a\not b(a-b)}=\frac{2b}{a(a-b)}$$

Before the beginning of tasks ,the **conditions** should be set. Since sharing with zero is not allowed :

$a\neq 0;\quad b\neq 0;\ a-b\neq 0\Rightarrow a\neq b$

2) **Simplify:** $\frac{1}{x^2-x}+\frac{2}{1-x^2}+\frac{1}{x^2+x}=$

$\frac{1}{x^2-x}+\frac{2}{1-x^2}+\frac{1}{x^2+x}=\frac{1}{x(x-1)}+\frac{2}{(1-x)(1+x)}+\frac{1}{x(x+1)}=$ what's the problem?

expressions $(1+x)$ and $(x+1)$ are not, because we have commutative law $(A+B=B+A)$,but express

$(x-1)$ and (1-x) **are**! This problem will be solved by law : A-B = - (B-A)

$$=\frac{1}{x(x-1)}\boxed{+}\frac{2}{(1-x)(1+x)}+\frac{1}{x(x+1)}$$
$$=\frac{1}{x(x-1)}-\frac{2}{(x-1)(1+x)}+\frac{1}{x(x+1)}=$$
$$=\frac{1\cdot(x+1)-2x+1(x-1)}{x(x-1)(x+1)}$$
$$=\frac{x+1-2x+x-1}{x(x-1)(x+1)}$$
$$=\frac{0}{x(x-1)(x+1)}=0$$

Of course, the task **conditions** are:

$x\neq 0;\ x-1\neq 0\Rightarrow x\neq 1;\ x+1\neq 0\Rightarrow x\neq -1$

3) $\frac{a+1}{a+2}+\frac{6a}{a^2-4}-\frac{2a-1}{a-2}$ =?

$$\frac{a+1}{a+2}+\frac{6a}{a^2-4}-\frac{2a-1}{a-2}=$$
$$\frac{a+1}{a+2}+\frac{6a}{(a-2)(a+2)}-\frac{2a-1}{a-2}=$$
$$\frac{(a+1)(a-2)+6a-(2a+1)(a+2)}{(a-2)(a+2)}=$$
$$\frac{(a^2-2a+a-2)+6a-(2a^2+4a-a-2)}{(a-2)(a+2)}=$$
$$\frac{a^2-2a+a-2+6a-2a^2-4a+a+2}{(a-2)(a+2)}=$$
$$\frac{-a^2+2a}{(a-2)(a+2)}=\frac{-a(a-2)}{(a-2)(a+2)}=\frac{-a}{a+2}$$

conditions of the task are: $a+2\neq 0\Rightarrow a\neq -2$
$a-2\neq 0\Rightarrow a\neq 2$

4) $\frac{x}{x-1}-\frac{3x-1}{x-2}+\frac{2x+1}{x^2-3x+2}=?$

$$\frac{x}{x-1}-\frac{3x-1}{x-2}+\frac{2x+1}{x^2-3x+2}=$$

First we do: $x^2-3x+2=x^2-2x-x+2=x(x-2)-1(x-2)=(x-2)(x-1)$

Now, we have:

$$\frac{x}{x-1}-\frac{3x-1}{x-2}+\frac{2x+1}{(x-2)(x-1)}=$$

$$\frac{x(x-2)-(3x-1)(x-1)+1(2x+1)}{(x-1)(x-2)}=$$

$$\frac{x^2-2x-(3x^2-3x-x+1)+2x+1}{(x-1)(x-2)}=$$

$$\frac{x^2-2x-3x^2+3x+x-1+2x+1}{(x-1)(x-2)}=$$

$$\frac{-2x^2+4x}{(x-1)(x-2)}=$$

$$\frac{-2x(x-2)}{(x-1)(x-2)}=\frac{-2x\,\cancel{(x-2)}}{(x-1)\,\cancel{(x-2)}}=\frac{-2x}{x-1}$$

conditions of the task are: $x-1\neq 0\Rightarrow x\neq 1$
$x-2\neq 0\Rightarrow x\neq 2$

5) $\frac{1}{x^2+10x+25}+\frac{1}{x^2-10x+25}+\frac{2}{x^2-25}=?$

$$\frac{1}{x^2+10x+25}+\frac{1}{x^2-10x+25}+\frac{2}{x^2-25}=$$

$$\frac{1}{(x+5)^2}+\frac{1}{(x-5)^2}+\frac{2}{(x-5)(x+5)}=$$

$$\frac{1\cdot(x-5)^2+1\cdot(x+5)^2+2\cdot(x^2-25)}{(x+5)^2(x-5)^2}=$$

$$\frac{x^2-10x+25+x^2+10x+25+2(x^2-25)}{(x+5)^2(x-5)^2}=$$

$$\frac{2x^2+50+2x^2-50}{(x+5)^2(x-5)^2}=$$

$$=\frac{4x^2}{(x+5)^2(x-5)^2}=\frac{4x^2}{(x^2-25)^2}$$

conditions of the task are: $x+5\neq 0\Rightarrow x\neq -5$
$x-5\neq 0\Rightarrow x\neq 5$

Multiplication and division of rational algebraic expression

We use: $\frac{A}{B}\cdot\frac{C}{D}=\frac{A\cdot C}{B\cdot D}$ and $\frac{A}{B}:\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}$

1) **Simplify**: $\frac{a^2-a}{a^2-1}\cdot\frac{a^2+2a+1}{a^2+a}=$

$$\frac{a^2-a}{a^2-1}\cdot\frac{a^2+2a+1}{a^2+a}=\frac{a(a-1)}{(a-1)(a+1)}\cdot\frac{(a+1)^2}{a(a+1)}==\frac{1}{1}\cdot\frac{1}{1}=1$$

conditions of the task are $a^2-1\neq 0$ and $a^2+a\neq 0$
$a\neq 1,\ a\neq -1\ ,a\neq 0$

2) **Simplify:** $\frac{a^2-ab}{a^2+ab}\cdot\frac{a^2b+ab^2}{ab}=?$

$$\frac{a^2-ab}{a^2+ab}\cdot\frac{a^2b+ab^2}{ab}=\frac{a(a-b)}{a(a+b)}\cdot\frac{ab(a+b)}{ab}=\frac{a-b}{1}=a-b \qquad a\neq 0,b\neq 0,a+b\neq 0$$

3) $\frac{x^2-25}{x^2-3x}:\frac{x^2+5x}{x^2-9}=?$

$$\frac{x^2-25}{x^2-3x}:\frac{x^2+5x}{x^2-9}=\frac{(x-5)(x+5)}{x(x-3)}:\frac{x(x+5)}{(x-3)(x+3)}=\frac{(x-5)(x+5)}{x(x-3)}\cdot\frac{(x-3)(x+3)}{x(x+5)}=\frac{(x-5)\cdot(x+3)}{x^2}$$

conditions of the task are: $x\neq 0, x-3\neq 0, x+3\neq 0$
$x\neq 3, x\neq -3$

4) $\frac{a^2+b^2}{1+2m+m^2}:\frac{a^4-b^4}{1-2m^2+m^4}$=?

$$\frac{a^2+b^2}{1+2m+m^2}:\frac{a^4-b^4}{1-2m^2+m^4}=$$
$$\frac{a^2+b^2}{(1+m)^2}:\frac{(a^2-b^2)(a^2+b^2)}{(1-m^2)^2}=$$
$$\frac{a^2+b^2}{(1+m)^2}\cdot\frac{(1-m)^2(1+m)^2}{(a-b)(a+b)(a^2+b^2)}=\frac{(1-m)^2}{(a-b)(a+b)}$$

$x\neq b, x\neq -b, m\neq 1,\ \ x\neq -1,$

5) $\dfrac{a^2+b^2-c^2+2ab}{a^2+c^2-b^2+2ac}$ =?

$$\frac{a^2+b^2-c^2+2ab}{a^2+c^2-b^2+2ac}=\frac{a^2+2ab+b^2-c^2}{a^2+2ac+c^2-b^2}=\frac{(a^2+b^2)-c^2}{(a+c)^2-b^2}=$$

$$\frac{(a+b-c)(a+b+c)}{(a+c-b)(a+c+b)}=\frac{(a+b-c)\,\cancel{(a+b+c)}}{(a+c-b)\,\cancel{(a+c+b)}}=\frac{a+b-c}{a+c-b}$$

$a+c-b\neq 0$ and $a+c+b\neq 0$

6) $\dfrac{x^2-5x+6}{x^2-3x+2}$ =?

$$\frac{x^2-5x+6}{x^2-3x+2}=\frac{x^2-3x-2x+6}{x^2-2x-x+2}=$$

$x-2\neq 0\,, x-1\neq 0$

$$=\frac{x(x-3)-2(x-3)}{x(x-2)-1(x-2)}=\frac{(x-3)(x-2)}{(x-2)(x-1)}=\frac{(x-3)\,\cancel{(x-2)}}{\cancel{(x-2)}\,(x-1)}=\frac{x-3}{x-1}$$

7) $\left(\dfrac{x}{y^2+xy}-\dfrac{2}{x+y}+\dfrac{y}{x^2+xy}\right):\left(\dfrac{x}{y}-2+\dfrac{y}{x}\right)=?$

$$\left(\frac{x}{y^2+xy}-\frac{2}{x+y}+\frac{y}{x^2+xy}\right):\left(\frac{x}{y}-2+\frac{y}{x}\right)=$$

$$\left(\frac{x}{y(y+x)}-\frac{2}{x+y}+\frac{y}{x(x+y)}\right):\left(\frac{x^2-2xy+y^2}{xy}\right)=$$

$$\frac{x^2-2xy+y^2}{xy(x+y)}:\frac{x^2-2xy+y^2}{xy}=$$

$$\frac{(x-y)^2}{xy(x+y)}\cdot\frac{xy}{(x-y)^2}=\frac{1}{x+y}$$

$x\neq 0,\;\; y\neq 0,\;\; x+y\neq 0,\;\; x-y\neq 0$

8) $\left(\frac{a}{6-3a}+\frac{a}{a+2}+\frac{4a}{a^2-4}\right):\frac{a-4}{a-2}=?$

$$\left(\frac{a}{3(2-a)}+\frac{a}{a+2}+\frac{4a}{(a-2)(a+2)}\right):\frac{a-4}{a-2}=$$

$$\left(\frac{a}{3(2-a)}+\frac{a}{a+2}+\frac{4a}{(a-2)(a+2)}\right):\frac{a-4}{a-2}=$$

$$\left(\frac{-a}{3(a-2)}+\frac{a}{a+2}+\frac{4a}{(a-2)(a+2)}\right)\cdot\frac{a-2}{a-4}=$$

$$\frac{-a(a+2)+3a(a-2)+12a}{3(a-2)(a+2)}\cdot\frac{a-2}{a-4}=$$

$$\frac{-a^2-2a+3a^2-6a+12a}{3(a+2)(a-4)}=$$

$$\frac{2a^2+4a}{3(a+2)(a-4)}=\frac{2a(a+2)}{3(a+2)(a-4)}=\frac{2a}{3(a-4)}$$

$a\neq 2,\ a\neq -2,\ a\neq 4$

9) $\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\frac{16}{1+x^{16}}=?$

This task can not be solved in the"classic" way, so we try to gather the first two:

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{1+x+1-x}{(1-x)(1+x)}=\frac{2}{1-x^2}$$

Now, we ad next : $\frac{2}{1+x^2}$

$$\frac{2}{1-x^2}+\frac{2}{1+x^2}=\frac{2(1+x^2)+2(1-x^2)}{(1-x^2)(1+x^2)}=\frac{2+2x^2+2-2x^2}{1-x^4}=\frac{4}{1-x^4}$$

This works!

$$\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{4+4x^2+4-4x^2}{(1-x^4)(1+x^2)}=\frac{8}{1-x^8}$$

Go further…

$$\frac{8}{1-x^8}+\frac{8}{1+x^8}=\frac{8+8x^8+8-8x^8}{(1-x^8)(1+x^8)}=\frac{16}{1-x^{16}}$$

Finally:

$$\frac{16}{1-x^{16}}+\frac{16}{1+x^{16}}=\frac{16+16x^{16}+16-16x^{16}}{(1-x^{16})(1+x^{16})}=\frac{32}{1-x^{32}}$$

$1-x\neq 0$ i $1+x\neq 0$

$$\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}+\frac{8}{1+x^8}+\frac{16}{1+x^{16}}=\frac{32}{1-x^{32}}$$

10) Prove that the value of expression $\dfrac{4}{a+\dfrac{1}{b+\dfrac{1}{c}}}:\dfrac{1}{a+\dfrac{1}{b}}-\dfrac{4}{b(abc+a+c)}$ **does not depend on *a, b, c* and *d*.**

$$\frac{4}{a+\frac{1}{b+\frac{1}{c}}}:\frac{1}{a+\frac{1}{b}}-\frac{4}{b(abc+a+c)}$$

$$\frac{4}{a+\frac{1}{b+\frac{1}{c}}}:\frac{1}{a+\frac{1}{b}}-\frac{4}{b(abc+a+c)}=$$

$$\frac{4}{a+\frac{1}{\frac{bc+1}{c}}}:\frac{1}{\frac{ab+1}{b}}-\frac{4}{b(abc+a+c)}=$$

<u>Use:</u> $\dfrac{\frac{A}{B}}{\frac{C}{D}}=\dfrac{AD}{BC}$

$$\frac{4}{a+\frac{c}{bc+1}}:\frac{b}{ab+1}-\frac{4}{b(abc+a+c)}=$$

$$\frac{4}{\frac{abc+a+c}{bc+1}}\cdot\frac{ab+1}{b}-\frac{4}{b(abc+a+c)}=$$

$$\frac{4(bc+1)}{abc+a+c}\cdot\frac{ab+1}{b}-\frac{4}{b(abc+a+c)}=$$

$$\frac{4(bc+1)(ab+1)}{b(abc+a+c)}-\frac{4}{b(abc+a+c)}=$$

$$\frac{4\left[(bc+1)(ab+1)-1\right]}{b(abc+a+c)}=$$

$$\frac{4\left[ab^2c+bc+ab+1-1\right]}{b(abc+a+c)}=\frac{4b(abc+c+a)}{b(abc+c+a)}=4$$

11) IF $a+b+c=0$ then $a^3+b^3+c^3=3abc$. Prove this!

$$\begin{aligned}
&a+b+c=0\\
&a+b=-c\\
&(a+b)^3=(-c)^3\\
&a^3+\underline{3a^2b}+\underline{3ab^2}+b^3=-c^3\\
&a^3+b^3+3ab(a+b)=-c^3 \rightarrow a+b=-c\\
&a^3+b^3-3abc=-c^3\\
&a^3+b^3+c^3=3abc
\end{aligned}$$

12) If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ then $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=-3$. Prove !

$$\frac{1}{a}+\frac{1}{b}=-\frac{1}{c}$$

$$\frac{b+a}{ab}=-\frac{1}{c}$$

$a+b=-\frac{ab}{c}$ / Devide with ***c*** to create a expression from the task

$$\boxed{\frac{a+b}{c}=-\frac{ab}{c^2}}$$

Similarlu, we have:

$$\frac{b+c}{a}=-\frac{bc}{a^2}$$

$$\frac{c+a}{b}=-\frac{ca}{b^2}$$

Now, back to the task:

$$\begin{aligned}
&\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}=\\
&-\frac{bc}{a^2}-\frac{ac}{b^2}-\frac{ab}{c^2}=\\
&-\frac{abc}{a^3}-\frac{abc}{b^3}-\frac{abc}{c^3}=\\
&=-abc\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)
\end{aligned}$$

We have to find this!

$$\frac{1}{a}+\frac{1}{b}=-\frac{1}{c}\,/()^3$$

$$\frac{1}{a^3}+3\cdot\frac{1}{a^2}\cdot\frac{1}{b}+3\cdot\frac{1}{a}\cdot\frac{1}{b^2}+\frac{1}{b^3}=-\frac{1}{c^3}$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{3}{ab}\left(\frac{1}{a}+\frac{1}{b}\right)=-\frac{1}{c^3}$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{3}{ab}\cdot\left(-\frac{1}{c}\right)=-\frac{1}{c^3}$$

$$\frac{1}{a^3}+\frac{1}{b^3}-\frac{3}{abc}=-\frac{1}{c^3}$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=+\frac{3}{abc}$$

$$=-abc\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

$$=-abc\cdot\frac{3}{abc}=-3$$

It is tricky……be careful!